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An extension to overpartitions of the Rogers–Ramanujan identities for even moduli

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Abstract

We study a class of well-poised basic hypergeometric series $\tilde{J}_{k,i}(a;x;q)$, interpreting these series as generating functions for overpartitions defined by multiplicity conditions on the number of parts. We also show how to interpret the $\tilde{J}_{k,i}(a;1;q)$ as generating functions for overpartitions whose successive ranks are bounded, for overpartitions that are invariant under a certain class of conjugations, and for special restricted lattice paths. We highlight the cases $(a,q)\to(1/q,q)$, $(1/q,q^2)$, and $(0,q)$, where some of the functions $\tilde{J}_{k,i}(a;1;q)$ become infinite products. The latter case corresponds to Bressoud's family of Rogers–Ramanujan identities for even moduli.

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1. Introduction

Over the years, a great amount of partition-theoretic information [2–5,10,12,17,19,23,25] has been extracted from Andrews' functions [9, Chapter 7] $J_{k,i}(a;x;q)$, which are defined by

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) + axq\,H_{k,i-1}(a;xq;q), \tag{1.1}$$

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where

$$H_{k,i}(a;x;q)=\sum_{n\geqslant 0}\frac{(-a)^n q^{kn^2+n-in}x^{kn}(1-x^i q^{2ni})(-1/a)_n(-axq^{n+1})_\infty}{(q)_n(xq^n)_\infty}. \tag{1.2}$$

Here and throughout we employ the usual basic hypergeometric series notations [21]

$$(a_1,a_2,\dots,a_j;q)_\infty=\prod_{m=0}^{\infty}\big(1-a_1q^m\big)\big(1-a_2q^m\big)\cdots\big(1-a_jq^m\big) \tag{1.3}$$

and

$$(a_1,a_2,\dots,a_j;q)_n=\frac{(a_1q^n,a_2q^n,\dots,a_jq^n;q)_\infty}{(a_1,a_2,\dots,a_j;q)_\infty}, \tag{1.4}$$

following the custom of dropping the base "; q" unless, of course, the base is something other than q.

Most recently [19], the first and third authors made a thorough combinatorial study of these functions, providing an interpretation of the general $J_{k,i}(a;x;q)$ in terms of overpartitions, which unified work of Andrews [5], Gordon [22], and the second author [23]. Moreover, it was shown that the $J_{k,i}(a;1;q)$ can be interpreted as generating functions for overpartitions with bounded successive ranks, for overpartitions with a specified Durfee dissection, and for certain restricted lattice paths. All of these interpretations generalized work of Andrews, Bressoud, and Burge on ordinary partitions [7,8,14–16].

In this paper we develop another class of functions which promise to be as fruitful as Andrews' $J_{k,i}(a;x;q)$. We call these $\tilde{J}_{k,i}(a;x;q)$ and they are defined by

$$\tilde{J}_{k,i}(a;x;q)=\tilde{H}_{k,i}(a;xq;q)+axq\tilde{H}_{k,i-1}(a;xq;q), \tag{1.5}$$

where

$$\tilde{H}_{k,i}(a;x;q)=\sum_{n\geqslant 0}\frac{(-a)^n q^{kn^2-\binom{n}{2}+n-in}x^{(k-1)n}(1-x^i q^{2ni})(-x,-1/a)_n(-axq^{n+1})_\infty}{(q^2;q^2)_n(xq^n)_\infty}. \tag{1.6}$$

This is not the first time these functions have appeared. The $\tilde{J}_{k,i}(a;x;q)$ are equal to the $F_{1,k,i}(-q,\infty;-1/a;x;q)$ introduced by Bressoud [13, Eq. (2.1)], and the $(-q)_\infty\tilde{H}_{k,i}(a;x;q)$ are equal to the functions $H_{k,i}(-1/a,-x;x;q)_2$ of Andrews [6]. However, neither of these two authors developed the analytic and combinatorial properties that we shall discover in Section 2.

Again the most natural combinatorial setting is that of overpartitions. Recall that an overpartition is a partition in which the first (or equivalently, final) occurrence of a number can be overlined [18]. Given an overpartition λ, let $f_\ell(\lambda)$ ($f_{\bar{\ell}}(\lambda)$) denote the number of occurrences of ℓ non-overlined (overlined) in λ. Let $V_\lambda(\ell)$ denote the number of overlined parts in λ less than or equal to ℓ. The following combinatorial interpretation of the general $\tilde{J}_{k,i}(a;x;q)$ is the principal result of the first half of this paper. Here and throughout the paper we assume that $k\geqslant 2$.

Theorem 1.1. *For $1 \leqslant i \leqslant k$ define the function $c_{k,i}(j,m,n)$ to be the number of overpartitions λ of n with m parts and j overlined parts such that* (i) $f_1(\lambda) + f_{\overline{1}}(\lambda) \leqslant i-1$, (ii) $f_\ell(\lambda) + f_{\ell+1}(\lambda) + f_{\overline{\ell+1}}(\lambda) \leqslant k-1$, *and* (iii) *if λ is saturated at ℓ, that is, if the maximum in* (ii) *is achieved, then* $\ell f_\ell(\lambda) + (\ell+1) f_{\ell+1}(\lambda) + (\ell+1) f_{\overline{\ell+1}}(\lambda) \equiv i - 1 + V_\lambda(\ell) \pmod 2$. *Then*

$$\tilde{J}_{k,i}(a;x;q) = \sum_{j,m,n \geqslant 0} c_{k,i}(j,m,n) a^j x^m q^n. \tag{1.7}$$

It turns out that the $\tilde{J}_{k,i}(a;1;q)$ are infinite products for $(a,q) = (0,q)$ and $(1/q, q^2)$, as well as for $(a,q) = (1/q, q)$ when $i = 1$, and hence we can deduce partition theorems from Theorem 1.1. In the case $(a,q) = (0,q)$, the product is

$$\tilde{J}_{k,i}(0;1;q) = \frac{(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty},$$

and we have recovered proof of Bressoud's Rogers–Ramanujan identities for even moduli [12]:

Corollary 1.2 *(Bressoud). For $1 \leqslant i \leqslant k-1$, let $\tilde{A}_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k$. Let $\tilde{B}_{k,i}(n)$ denote the number of partitions λ of n such that* (i) $f_1(\lambda) \leqslant i-1$, (ii) $f_\ell(\lambda) + f_{\ell+1}(\lambda) \leqslant k-1$, *and* (iii) *if $f_\ell(\lambda) + f_{\ell+1}(\lambda) = k-1$, then $\ell f_\ell(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv i-1 \pmod 2$. Then $\tilde{A}_{k,i}(n) = \tilde{B}_{k,i}(n)$.*

When $(a,q) = (1/q, q^2)$, the product is

$$\tilde{J}_{k,i}\big(1/q;1;q^2\big) = \frac{(-q;q^2)_\infty (q^{2i-1}, q^{4k-2i-1}, q^{4k-2}; q^{4k-2})_\infty}{(q^2;q^2)_\infty},$$

and the result is Bressoud's mod $4k-2$ companion [13, Eq. (3.9) and Theorem 2] to Andrews' generalization of the Göllnitz–Gordon identities [5]:

Corollary 1.3. *For $1 \leqslant i \leqslant k-1$, let $\tilde{A}^2_{k,i}(n)$ denote the number of partitions of n where even parts are multiples of 4 not divisible by $8k-4$ and odd parts are not congruent to $\pm(2i-1)$ modulo $4k-2$, with parts congruent to $2k-1$ modulo $4k-2$ not repeatable. Let $\tilde{B}^2_{k,i}(n)$ denote the number of partitions λ of n such that* (i) $f_1(\lambda) + f_2(\lambda) \leqslant i-1$, (ii) $f_{2\ell}(\lambda) + f_{2\ell+1}(\lambda) + f_{2\ell+2}(\lambda) \leqslant k-1$, *and* (iii) *if the maximum in* (ii) *is achieved at ℓ, then $\ell f_{2\ell}(\lambda) + (\ell+1) f_{2\ell+2}(\lambda) + (\ell+1) f_{2\ell+1}(\lambda) \equiv i - 1 + V^o_\lambda(\ell) \pmod 2$. (Here $V^o_\lambda(\ell)$ is the number of odd parts of λ less than 2ℓ). Then $\tilde{A}^2_{k,i}(n) = \tilde{B}^2_{k,i}(n)$.*

Finally, when $(a,q) = (1/q, q)$ and $i = 1$, the product is

$$\tilde{J}_{k,1}(1/q;1;q) = \frac{(-q)_\infty (q, q^{2k-2}, q^{2k-1}; q^{2k-1})_\infty}{(q)_\infty},$$

and the result is an odd modulus companion to Theorem 1.2 of [23].

Corollary 1.4. *Let $\tilde{A}^3_k(n)$ denote the number of overpartitions whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k-1$. Let $\tilde{B}^3_k(n)$ denote the number of overpartitions λ of n such* *that* (i) $f_1(\lambda) = 0$, (ii) $f_\ell(\lambda) + f_{\bar{\ell}}(\lambda) + f_{\ell+1}(\lambda) \leqslant k-1$, *and* (iii) *if the maximum in condition* (ii) *is achieved at* ℓ, *then* $\ell f_\ell(\lambda) + \ell f_{\bar{\ell}}(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv V_\lambda(\ell) \pmod 2$. *Then* $\tilde{A}_k^3(n) = \tilde{B}_k^3(n)$.

In the second half of the paper, we discuss three more combinatorial interpretations of the $\tilde{J}_{k,i}(a;1;q)$: one involving the theory of successive ranks for overpartitions as developed in [19], one involving a generalization to overpartitions of Garvan's k-conjugation for partitions [20], and one involving a generalization of some lattice paths of Bressoud and Burge [14–16]. The following is the main theorem of this part, the combinatorial concepts being necessarily fully defined later in the paper. When $a = 0$ and $X = C$, D, or E, we recover some of the main results of [14–16].

Theorem 1.5.

- *Let* $\tilde{B}_{k,i}(n,j)$ *denote the number of overpartitions* λ *of* n *which are counted by* $c_{k,i}(j,m,n)$ *for some* m.
- *Let* $\tilde{C}_{k,i}(n,j)$ *denote the number of overpartitions of* n *with* j *overlined parts whose successive ranks lie in* $[-i+2, 2k-i-2]$.
- *Let* $\tilde{D}_{k,i}(n,j)$ *denote the number of self-*(k,i)*-conjugate overpartitions of* n *with* j *overlined parts.*
- *Let* $\tilde{E}_{k,i}(n,j)$ *denote the number of special lattice paths of major index* n *with* j *South steps which start at* $k-i$, *whose height is less than* k *and where the peaks of coordinates* $(x, k-1)$ *are such that* $x-u$ *is congruent to* $i-1$ *modulo* 2 (*u is the number of South steps to the left of the peak*).

Then for $X = B$, C, D, *or* E,

$$\sum_{n,j\geqslant 0} \tilde{X}_{k,i}(n,j) a^j q^n = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1/a)_n (-1)^n a^n q^{(2k-1)\binom{n+1}{2} - in + n}}{(-aq)_n}. \tag{1.8}$$

Again, the right-hand side of (1.8) is in many cases an infinite product, and hence there are results like Corollaries 1.2–1.4 involving the functions $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$. However, we shall not highlight these corollaries.

The paper is organized as follows. In the next section we study the basic properties of the $\tilde{J}_{k,i}(a;x;q)$ and give proofs of Theorem 1.1 and Corollaries 1.2–1.4. In Section 3, we compute the generating function of the paths counted by $\tilde{E}_{k,i}(n,j)$ to show that they are in bijection with the overpartitions counted by $\tilde{B}_{k,i}(n,j)$. In Section 4, we present a direct bijection between the paths counted by $\tilde{E}_{k,i}(n,j)$ and the overpartitions counted by $\tilde{C}_{k,i}(n,j)$. In Section 5, we compute the generating function of the overpartitions counted by $\tilde{D}_{k,i}(n,j)$ and use the Bailey lattice to show that they are in bijection with the paths counted by $\tilde{E}_{k,i}(n,j)$. We conclude in Section 6 with some suggestions for future research.

2. The $\tilde{J}_{k,i}(a;x;q)$

We begin by proving some facts about the functions $\tilde{H}_{k,i}(a;x;q)$ and $\tilde{J}_{k,i}(a;x;q)$ defined in the introduction.

Lemma 2.1.

$$\tilde{H}_{k,0}(a;x;q)=0, \tag{2.1}$$

$$\tilde{H}_{k,-i}(a;x;q)=-x^{-i}\tilde{H}_{k,i}(a;x;q), \tag{2.2}$$

$$\tilde{H}_{k,i}(a;x;q)-\tilde{H}_{k,i-2}(a;x;q)=x^{i-2}(1+x)\tilde{J}_{k,k-i+1}(a;x;q). \tag{2.3}$$

Proof. The first part is trivial and the second part follows from the fact that

$$-x^{-i}q^{-in}\big(1-x^iq^{2ni}\big)=q^{-n(-i)}\big(1-x^{-i}q^{2n(-i)}\big).$$

For the third part, we have

$$\begin{aligned}
&\tilde{H}_{k,i}(a;x;q)-\tilde{H}_{k,i-2}(a;x;q)\\
&\quad=\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+n}x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty}{(q^2;q^2)_n(xq^n)_\infty}\\
&\qquad\times\big(q^{-in}-x^iq^{in}-q^{(2-i)n}+\big(xq^n\big)^{i-2}\big)\\
&\quad=\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+n}x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty q^{-in}(1-q^{2n})}{(q^2;q^2)_n(xq^n)_\infty}\\
&\qquad+\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+n}x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty x^{i-2}q^{n(i-2)}(1-x^2q^{2n})}{(q^2;q^2)_n(xq^n)_\infty}\\
&\quad=\sum_{n\geqslant 1}\frac{(-a)^nq^{kn^2-\binom{n}{2}+n}x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty q^{-in}}{(q^2;q^2)_{n-1}(xq^n)_\infty}\\
&\qquad+\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+1})_\infty x^{i-2}q^{n(i-2)}}{(q^2;q^2)_n(xq^{n+1})_\infty}\\
&\quad=\sum_{n\geqslant 0}\frac{(-a)^{n+1}q^{kn^2+2kn+k-\binom{n+1}{2}+n+1}x^{kn+k-n-1}(-x,-1/a)_{n+1}(-axq^{n+2})_\infty q^{-in-i}}{(q^2;q^2)_n(xq^{n+1})_\infty}\\
&\qquad+\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+1})_\infty x^{i-2}q^{n(i-2)}}{(q^2;q^2)_n(xq^{n+1})_\infty}\\
&\quad=x^{i-2}\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+ni-n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}\\
&\qquad\times\big(\big(1+axq^{n+1}\big)-ax^{k-i+1}q^{2kn-2ni+n+k-i+1}\big(1+q^n/a\big)\big)\\
&\quad=x^{i-2}\sum_{n\geqslant 0}\frac{(-a)^nq^{kn^2-\binom{n}{2}+ni-n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}
\end{aligned}$$

$$\times \left(1 - x^{k-i+1} q^{(k-i+1)(2n+1)}\right)$$
$$+ x^{i-2} \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+ni-n} x^{(k-1)n} (-x)_{n+1} (-1/a)_n (-axq^{n+2})_\infty}{(q^2;q^2)_n (xq^{n+1})_\infty}$$
$$\times axq^{n+1}\left(1 - x^{k-i} q^{(k-i)(2n+1)}\right)$$
$$= x^{i-2}(1+x) \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+ni-n} x^{(k-1)n} (-xq)_n (-1/a)_n (-axq^{n+2})_\infty}{(q^2;q^2)_n (xq^{n+1})_\infty}$$
$$\times \left(1 - x^{k-i+1} q^{(k-i+1)(2n+1)}\right)$$
$$+ x^{i-2}(1+x)axq \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+ni} x^{(k-1)n} (-xq)_n (-1/a)_n (-axq^{n+2})_\infty}{(q^2;q^2)_n (xq^{n+1})_\infty}$$
$$\times \left(1 - x^{k-i} q^{(k-i)(2n+1)}\right)$$
$$= x^{i-2}(1+x)\left(\tilde{H}_{k,k-i+1}(a;xq;q) + axq\tilde{H}_{k,k-i}(a;xq;q)\right)$$
$$= x^{i-2}(1+x)\tilde{J}_{k,k-i+1}(a;xq;q). \qquad \square$$

Now assume that $1 \leqslant i \leqslant k$. The following recurrences for the $\tilde{J}_{k,i}(a;x;q)$ are fundamental.

Theorem 2.2.

$$\tilde{J}_{k,1}(a;x;q) = \tilde{J}_{k,k}(a;xq;q), \tag{2.4}$$
$$\tilde{J}_{k,2}(a;x;q) = (1+xq)\tilde{J}_{k,k-1}(a;xq;q) + axq\tilde{J}_{k,k}(a;xq;q), \tag{2.5}$$
$$\begin{aligned}\tilde{J}_{k,i}(a;x;q) - \tilde{J}_{k,i-2}(a;x;q) &= (xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+1}(a;xq;q)\\ &\quad + a(xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+2}(a;xq;q) \quad (3 \leqslant i \leqslant k).\end{aligned} \tag{2.6}$$

Proof. Using (2.3) followed by (2.2) and then (2.1), we have

$$\begin{aligned}\tilde{J}_{k,k}(a;xq;q) &= \frac{\tilde{H}_{k,1}(a;xq;q) - \tilde{H}_{k,-1}(a;xq;q)}{(xq)^{-1}(1+xq)}\\ &= \frac{\tilde{H}_{k,1}(a;xq;q) + (xq)^{-1}\tilde{H}_{k,1}(a;xq;q)}{(xq)^{-1}(1+xq)}\\ &= \tilde{H}_{k,1}(a;xq;q)\\ &= \tilde{H}_{k,1}(a;xq;q) + axqH_{k,0}(a;xq;q)\\ &= \tilde{J}_{k,1}(a;x;q),\end{aligned}$$

which is (2.4). For (2.5), we have

$$\begin{aligned}\tilde{J}_{k,2}(a;xq;q) &= \tilde{H}_{k,2}(a;xq;q) + axq\tilde{H}_{k,1}(a;xq;q)\\ &= \tilde{H}_{k,2}(a;xq;q) - \tilde{H}_{k,0}(a;xq;q) + axq\tilde{H}_{k,1}(a;xq;q)\\ &= (1+xq)\tilde{J}_{k,k-1}(a;xq;q) + axq\tilde{J}_{k,1}(a;x;q).\end{aligned}$$

Finally, using (2.3) we have

$$\begin{aligned}\tilde{J}_{k,i}(a;x;q)-\tilde{J}_{k,i-2}(a;x;q)&=\tilde{H}_{k,i}(a;xq;q)+axq\tilde{H}_{k,i-1}(a;xq;q)\\&\quad-\tilde{H}_{k,i-2}(a;xq;q)-axq\tilde{H}_{k,i}(a;xq;q)\\&=(xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+1}(a;xq;q)\\&\quad+axq(xq)^{i-3}(1+xq)\tilde{J}_{k,k-i+2}(a;xq;q),\end{aligned}$$

which is (2.6) and which completes the proof of the theorem. □

We now turn to the proof of Theorem 1.1. If we write

$$\tilde{J}_{k,i}(a;x;q)=\sum_{j,m,n\geqslant 0}b_{k,i}(j,m,n)a^jx^mq^n,$$

then the recurrences in Theorem 2.2 imply that

$$b_{k,1}(j,m,n)=b_{k,k}(j,m,n-m), \tag{2.7}$$

$$\begin{aligned}&b_{k,2}(j,m,n)\\&\quad=b_{k,k-1}(j,m,n-m)+b_{k,k-1}(j,m-1,n-m)+b_{k,k}(j-1,m-1,n-m),\end{aligned} \tag{2.8}$$

and for $3\leqslant i\leqslant k$,

$$\begin{aligned}&b_{k,i}(j,m,n)-b_{k,i-2}(j,m,n)\\&\quad=b_{k,k-i+1}(j,m-i+2,n-m)+b_{k,k-i+1}(j,m-i+1,n-m)\\&\qquad+b_{k,k-i+2}(j-1,m-i+2,n-m)+b_{k,k-i+2}(j-1,m-i+1,n-m).\end{aligned} \tag{2.9}$$

We shall demonstrate that the $c_{k,i}(j,m,n)$ also satisfy these recurrences. In what follows we shall repeatedly employ a mapping $\lambda\to\hat{\lambda}$, where $\hat{\lambda}$ is obtained by removing all of the ones from λ and then subtracting one from each remaining part, or equivalently, by removing the first column from the Ferrer's diagram of λ. Before continuing, we make a couple of observations regarding this mapping. First, if λ satisfies condition (ii) in the statement of the theorem, so does $\hat{\lambda}$. Second, if λ is an overpartition counted by $c_{k,i}(j,m,n)$ and $\hat{\lambda}$ is saturated at ℓ, then λ was saturated at $\ell+1$, so we have

$$\begin{aligned}&\ell f_\ell(\hat{\lambda})+(\ell+1)f_{\ell+1}(\hat{\lambda})+(\ell+1)f_{\overline{\ell+1}}(\hat{\lambda})\\&\quad=\ell f_{\ell+1}(\lambda)+(\ell+1)f_{\ell+2}(\lambda)+(\ell+1)f_{\overline{\ell+2}}(\lambda)\\&\quad=(\ell+1)f_{\ell+1}(\lambda)+(\ell+2)f_{\ell+2}(\lambda)+(\ell+2)f_{\overline{\ell+2}}(\lambda)-\big(f_\ell(\hat{\lambda})+f_{\ell+1}(\hat{\lambda})+f_{\overline{\ell+1}}(\hat{\lambda})\big)\\&\quad\equiv i-1+V_\lambda(\ell+1)-\big(f_\ell(\hat{\lambda})+f_{\ell+1}(\hat{\lambda})+f_{\overline{\ell+1}}(\hat{\lambda})\big)\pmod 2\\&\quad\equiv V_\lambda(\ell+1)+k-i\pmod 2.\end{aligned} \tag{2.10}$$

Finally, it is clear that

$$V_{\hat{\lambda}}(\ell) \equiv \begin{cases} V_\lambda(\ell+1) \pmod 2, & \text{if } \overline{1} \notin \lambda, \\ V_\lambda(\ell+1)+1 \pmod 2, & \text{if } \overline{1} \in \lambda. \end{cases} \tag{2.11}$$

We begin with (2.7). Given an overpartition λ counted by $c_{k,1}(j,m,n)$, $\hat{\lambda}$ is an overpartition of $n-m$ with m parts, j of which are overlined. Since λ could have had at most $k-1$ twos, $\hat{\lambda}$ has at most $k-1$ ones. If $\hat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\hat{\lambda}) + (\ell+1)f_{\ell+1}(\hat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\hat{\lambda}) \equiv k-1+V_{\hat{\lambda}}(\ell) \pmod 2$. Thus $\hat{\lambda}$ is an overpartition counted by $c_{k,k}(j,m,n-m)$. Since the mapping from λ to $\hat{\lambda}$ is reversible, we have the recurrence (2.7) for the functions $c_{k,i}(j,m,n)$.

We turn to (2.8). Suppose now that λ is an overpartition counted by $c_{k,2}(j,m,n)$. Then λ has at most one 1. We consider three cases.

First, if λ has no ones, then it can have at most $k-2$ twos. For if λ had $k-1$ twos, then $1f_1(\lambda)+2f_2(\lambda)+2f_{\overline{2}}(\lambda) \equiv 0 \pmod 2$ violates condition (iii) in the definition of the $c_{k,2}(j,m,n)$. Hence $\hat{\lambda}$ is an overpartition of $n-m$ into m parts, j of which are overlined, and having at most $k-2$ ones. If $\hat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\hat{\lambda}) + (\ell+1)f_{\ell+1}(\hat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\hat{\lambda}) \equiv k-2+V_{\hat{\lambda}}(\ell) \pmod 2$. Hence $\hat{\lambda}$ is an overpartition counted by $c_{k,k-1}(j,m,n-m)$.

Second, if 1 occurs (non-overlined) in λ, then there can be up to $k-2$ twos, so $\hat{\lambda}$ has at most $k-2$ ones. If $\hat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\hat{\lambda}) + (\ell+1)f_{\ell+1}(\hat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\hat{\lambda}) \equiv k-2+V_{\hat{\lambda}}(\ell) \pmod 2$. Hence $\hat{\lambda}$ is an overpartition counted by $c_{k,k-1}(j,m-1,n-m)$.

Third and finally, if $\overline{1}$ occurs in λ, then there can be at most $k-1$ twos, so $\hat{\lambda}$ has at most $k-1$ ones. If $\hat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\hat{\lambda}) + (\ell+1)f_{\ell+1}(\hat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\hat{\lambda}) \equiv k-1+V_{\hat{\lambda}}(\ell) \pmod 2$. Hence $\hat{\lambda}$ is an overpartition counted by $c_{k,k}(j-1,m-1,n-m)$.

Since the mappings are reversible, we have the recurrence (2.8) for the functions $c_{k,i}(j,m,n)$.

For the recurrence (2.9), everything continues to work out nicely as above. Note that for $3 \leqslant i \leqslant k$, $c_{k,i}(j,m,n) - c_{k,i-2}(j,m,n)$ counts those overpartitions λ counted by $c_{k,i}(j,m,n)$ having exactly $i-1$ or $i-2$ ones. We consider two cases. First, if $\overline{1}$ does not occur, then if λ has $i-1$ ones then there can be at most $k-i$ twos in λ and therefore at most $k-i$ ones in $\hat{\lambda}$. If λ has $i-2$ ones there can still be at most $k-i$ twos, or else the defining condition (iii) would be violated. So in either case, there are at most $k-i$ ones in $\hat{\lambda}$. And, in either case, if $\hat{\lambda}$ is saturated at ℓ, using (2.10) and (2.11) as usual shows that $\ell f_\ell(\hat{\lambda}) + (\ell+1)f_{\ell+1}(\hat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\hat{\lambda}) \equiv k-i+V_{\hat{\lambda}}(\ell) \pmod 2$. So $\hat{\lambda}$ is an overpartition counted by $c_{k,k-i+1}(j,m-i+1,n-m)$ in the first case, and $c_{k,k-i+1}(j,m-i+2,n-m)$ in the second case.

Now if $\overline{1}$ does occur in λ, then whether there are $i-1$ or $i-2$ ones there can be up to $k-i+1$ twos, and so $\hat{\lambda}$ has at most $k-i+1$ ones. Finally, if $\hat{\lambda}$ is saturated at ℓ, then $\ell f_\ell(\hat{\lambda}) + (\ell+1)f_{\ell+1}(\hat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\hat{\lambda}) \equiv k-i+1+V_{\hat{\lambda}}(\ell) \pmod 2$. Therefore $\hat{\lambda}$ is an overpartition counted by $c_{k,k-i+2}(j-1,m-i+1,n-m)$ if λ has $i-1$ ones and $c_{k,k-i+2}(j-1,m-i+2,n-m)$ if λ has $i-2$ ones. Again the mappings here are reversible, so we have the recurrence (2.9) for the functions $c_{k,i}(j,m,n)$.

To finalize the claim that the two families of functions are equal, we note that

$$b_{k,i}(j,m,n) = \begin{cases} 0, & \text{if } j<0, m \leqslant 0 \text{ or } n \leqslant 0, \text{ and } (j,m,n) \neq (0,0,0), \\ 1, & \text{if } (j,m,n) = (0,0,0), \end{cases} \tag{2.12}$$

which is indeed also true for the $c_{k,i}(j,m,n)$.

Before deducing Corollaries 1.2–1.4, we will prove a proposition which is a piece of Theorem 1.5 and from which it follows that several instances of the $\tilde{J}_{k,i}(a;1;q)$ are infinite products.

Proposition 2.3. *We have*

$$\tilde{J}_{k,i}(a;1;q) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1/a)_n(-1)^n a^n q^{(2k-1)\binom{n+1}{2}-in+n}}{(-aq)_n}. \tag{2.13}$$

Proof. Using the definition, we have

$$\begin{aligned}
\tilde{J}_{k,i}(a;1;q) &= \tilde{H}_{k,i}(a;q;q) + aq\tilde{H}_{k,i-1}(a;q;q) \\
&= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in}(1-q^{i(2n+1)})(-1/a)_n}{(-aq)_{n+1}} \\
&\quad + aq\frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-(i-1)n}(1-q^{(i-1)(2n+1)})(-1/a)_n}{(-aq)_{n+1}} \\
&= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in}(-1/a)_n}{(-aq)_{n+1}} \\
&\quad \times \left(1-q^{(2n+1)i}+aq^{n+1}-aq^{n+1+(i-1)(2n+1)}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in}(-1/a)_n}{(-aq)_n} \\
&\quad - \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in+i(2n+1)}(-1/a)_n}{(-aq)_{n+1}}\left(1+aq^{-n}\right).
\end{aligned}$$

In this last sum, we replace n by $-n-1$ and simplify using the fact that

$$(x)_{-n} = \frac{(-1)^n q^{\binom{n+1}{2}}}{x^n(q/x)_n}.$$

The result is precisely (2.13). $\square$

Corollary 2.4. *We have*

$$\tilde{J}_{k,i}(0;1;q) = \frac{(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty}, \tag{2.14}$$

$$\tilde{J}_{k,i}\left(1/q;1;q^2\right) = \frac{(-q;q^2)_\infty(q^{2i-1},q^{4k-2i-1},q^{4k-2};q^{4k-2})_\infty}{(q^2;q^2)_\infty}, \tag{2.15}$$

and

$$\tilde{J}_{k,1}(1/q,1;q) = \frac{(-q)_\infty (q,q^{2k-2},q^{2k-1};q^{2k-1})_\infty}{(q)_\infty}. \tag{2.16}$$

Proof. These follow easily from Proposition 2.3 and the Jacobi triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-zq,-1/z,q)_\infty. \tag{2.17}$$

Indeed

$$\tilde{J}_{k,i}(0;1;q) = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-in} = \frac{(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty},$$

using Eq. (2.17) with $q \to q^{2k}$ and $z \to -q^{-i}$. Similarly,

$$\begin{aligned}\tilde{J}_{k,i}\big(1/q;1;q^2\big) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(4k-2)\binom{n+1}{2}-(2i-1)n}\\ &= \frac{(-q;q^2)_\infty(q^{2i-1},q^{4k-2i-1},q^{4k-2};q^{4k-2})_\infty}{(q^2;q^2)_\infty},\end{aligned}$$

using Eq. (2.17) with $q \to q^{4k-2}$ and $z \to -q^{-(2i-1)}$. Finally,

$$\begin{aligned}\tilde{J}_{k,1}(1/q,1;q) &= \frac{(-1)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{1+q^n}{2}(-1)^n q^{(2k-1)\binom{n+1}{2}-n}\\ &= \frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\big(q^{(2k-1)\binom{n+1}{2}-n}+q^{(2k-1)\binom{n+1}{2}}\big)\\ &= \frac{(-q)_\infty((q,q^{2k-2},q^{2k-1};q^{2k-1})_\infty+(1,q^{2k-1},q^{2k-1};q^{2k-1})_\infty)}{(q)_\infty},\end{aligned}$$

using Eq. (2.17) with $q \to q^{2k-2}$ and $z \to -q^{-1}$ or -1. □

We are now ready to prove Corollaries 1.2–1.4. In the following, λ is an overpartition of n with j overlined parts, and hence it is counted in the coefficient of $q^n a^j$ in $\tilde{J}_{k,i}(a,1;q)$. This overpartition is such that (i) $f_1(\lambda)+f_{\bar{1}}(\lambda) \leqslant i-1$, (ii) $f_\ell(\lambda)+f_{\ell+1}(\lambda)+f_{\overline{\ell+1}}(\lambda) \leqslant k-1$, and (iii) if λ is saturated at ℓ, that is, if the maximum in (ii) is achieved, then $\ell f_\ell(\lambda)+(\ell+1)f_{\ell+1}(\lambda)+(\ell+1)f_{\overline{\ell+1}}(\lambda) \equiv i-1+V_\lambda(\ell) \pmod 2$.

For Corollary 1.2, we consider the functions $\tilde{J}_{k,i}(0;1;q)$. From Theorem 1.1 we easily see that the coefficient of q^n in $\tilde{J}_{k,i}(0;1;q)$ is $\tilde{B}_{k,i}(n)$. Indeed when $(a,q)=(0,q)$, this implies that λ has no overlined parts, that is $f_{\bar{\ell}} = V_\lambda(\ell) = 0$ for all ℓ. Therefore the conditions (i), (ii) and (iii) are now (i) $f_1(\lambda) \leqslant i-1$, (ii) $f_\ell(\lambda)+f_{\ell+1}(\lambda) \leqslant k-1$, and (iii) if the maximum in (ii) is achieved at ℓ, then $\ell f_\ell(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv i-1 \pmod 2$. On the other hand, from (2.14) of Corollary 2.4, the coefficient of q^n in $\tilde{J}_{k,i}(0;1;q)$ is also $\tilde{A}_{k,i}(n)$.

For Corollary 1.3, we use the functions $\tilde{J}_{k,i}(1/q;1;q^2)$. A little thought reveals that the coefficient of q^n in $\tilde{J}_{k,i}(1/q;1;q^2)$ is $\tilde{B}^2_{k,i}(n)$. When $(a,q) = (1/q, q^2)$, for any ℓ all the parts equal to $\bar{\ell}$ in λ are changed to $2\ell-1$ and all the parts equal to ℓ in λ are changed to 2ℓ. This implies that (i) $f_1(\lambda) + f_2(\lambda) \leqslant i-1$, (ii) $f_{2\ell}(\lambda) + f_{2\ell+1}(\lambda) + f_{2\ell+2}(\lambda) \leqslant k-1$, and (iii) if the maximum in (ii) is achieved at ℓ, then $\ell f_{2\ell}(\lambda) + (\ell+1) f_{2\ell+2}(\lambda) + (\ell+1) f_{2\ell+1}(\lambda) \equiv i-1+V^o_\lambda(\ell) \pmod 2$.

Rewriting of the product in (2.15) as

$$\left(q^2;q^4\right)_\infty \left(q^{8k-4};q^{8k-4}\right)_\infty \left(q^{2i-1},q^{4k-2i-1};q^{4k-2}\right)_\infty \left(-q^{2k-1};q^{4k-2}\right)_\infty \times \prod_{n \not\equiv 2k-1 \pmod{4k-2}} \frac{1}{(1-q^n)}$$

shows that the coefficient of q^n in $\tilde{J}_{k,i}(1/q;1;q^2)$ is also $\tilde{A}^2_{k,i}(n)$.

Finally, for Corollary 1.4, we use the functions $\tilde{J}_{k,1}(1/q;1;q)$. Again it may readily be seen that the coefficient of q^n therein is $\tilde{B}^3_k(n)$. Indeed when $i=1$, $f_1(\lambda) = f_{\bar{1}}(\lambda) = 0$, and when $(a,q) = (1/q, q)$ all the overlined parts of λ are decreased by one. That implies that (i) $f_1(\lambda) = 0$, (ii) $f_\ell(\lambda) + f_{\bar{\ell}}(\lambda) + f_{\ell+1}(\lambda) \leqslant k-1$, and (iii) if the maximum in condition (ii) is achieved at ℓ, then $\ell f_\ell(\lambda) + (\ell+1) f_{\bar{\ell}}(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv V_\lambda(\ell-1) \pmod 2$. As $V_\lambda(\ell-1) + f_{\bar{\ell}}(\lambda) = V_\lambda(\ell)$, this is equivalent to $\ell f_\ell(\lambda) + \ell f_{\bar{\ell}}(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv V_\lambda(\ell) \pmod 2$. On the other hand, from (2.16) of Corollary 2.4, the coefficient q^n in $\tilde{J}_{k,1}(1/q;1;q)$ is also $\tilde{A}^3_k(n)$.

3. Lattice paths

In this section we define the special lattice paths and establish Theorem 1.5 for $X = E$. We study paths in the first quadrant that use four kinds of unitary steps:

- North-East NE: $(x,y) \to (x+1, y+1)$,
- South-East SE: $(x,y) \to (x+1, y-1)$,
- South S: $(x,y) \to (x, y-1)$,
- East E: $(x,0) \to (x+1, 0)$.

The *height* corresponds to the y-coordinate of a vertex. A South step can only appear after a North-East step and an East step can only appear at height 0. The paths must end on the x-axis with a South-East or South step. A *peak* is a vertex preceded by a North-East step and followed by a South step (in which case it will be called a *NES peak*) or by a South-East step (in which case it will be called a *NESE peak*). The *major index* of a path is the sum of the x-coordinates of its peaks (see Fig. 1 for example). When the paths have no South steps, this is the definition of the paths in [14].

Let i be a positive integer with $i \leqslant k$. Let $\tilde{E}_{k,i}(n,j)$ be the number of paths of major index n with j South steps which satisfy the following *special* (k,i)*-conditions*: (i) the paths start at height $k-i$, (ii) their height is less than k, (iii) every peak of coordinates $(x, k-1)$ satisfies $x - u \equiv i-1 \pmod 2$ where u is the number of South steps to the left of the peak.

Let $\tilde{\mathcal{E}}_{k,i}(a,q)$ be the generating function of these paths, $\tilde{\mathcal{E}}_{k,i}(a,q) = \sum_{n,j} \tilde{E}_{k,i}(n,j) a^j q^n$. Let $\tilde{\mathcal{E}}_{k,i}(N)$ be the generating function of paths counted by $\tilde{\mathcal{E}}_{k,i}(a,q)$ which have N peaks. Moreover,

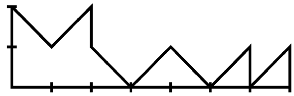

Fig. 1. This path has four peaks: three NES peaks (located at $(2,2)$, $(6,1)$ and $(7,1)$) and one NESE peak (located at $(4,1)$). Its major index is $2+4+6+7=19$.

for $0 \leqslant i < k$, let $\tilde{\Gamma}_{k,i}(N)$ be the generating function of paths obtained by deleting the first NE step of a path which is counted in $\tilde{\mathcal{E}}_{k,i+1}(N)$ and starts with a NE step. We begin our computation of these generating functions with some simple recurrences and initial conditions.

Proposition 3.1.

$$\tilde{\mathcal{E}}_{k,i}(N) = q^N \tilde{\mathcal{E}}_{k,i+1}(N) + q^N \tilde{\Gamma}_{k,i-1}(N), \quad 0 < i < k, \tag{3.1}$$

$$\tilde{\Gamma}_{k,i}(N) = q^N \tilde{\Gamma}_{k,i-1}(N) + \left(a + q^{N-1}\right)\tilde{\mathcal{E}}_{k,i+1}(N-1), \quad 0 < i < k, \tag{3.2}$$

$$\tilde{\mathcal{E}}_{k,k}(N) = q^N \tilde{\mathcal{E}}_{k,k-1}(N) + q^N \tilde{\Gamma}_{k,k-1}(N), \tag{3.3}$$

$$\tilde{\mathcal{E}}_{k,i}(0) = 1, \tag{3.4}$$

$$\tilde{\Gamma}_{k,0}(N) = 0. \tag{3.5}$$

Proof. First, there is a unique path with no peaks. This gives $\tilde{\mathcal{E}}_{k,i}(0) = 1$, which is (3.4). The initial condition (3.5) is just as straightforward, for a path that starts at height $k-1$ cannot start with a North-East step.

Now, if the path has at least one peak, then we may take off its first step and shift the path one unit to the left. If $0 < i < k$, then a path counted by $\tilde{\mathcal{E}}_{k,i}(N)$ starts with a North-East step (corresponding to $q^N \Gamma_{k,i-1}(N)$) or a South-East step (corresponding to $q^N \mathcal{E}_{k,i+1}(N)$). This gives (3.1). (Notice that when we remove the first NE or SE step, we increase or decrease i by 1 and thus change the parity of $i-1$; moreover, all the peaks are shifted by 1, so the parity of $x-u-i$ is not changed and condition (iii) in the definition of the special (k,i)-conditions is respected.)

For (3.2), $\tilde{\Gamma}_{k,i}(N)$ is the generating function for the paths counted by $\tilde{\mathcal{E}}_{k,i+1}(N)$ where the first North-East step was deleted. These paths can start with a North-East step ($q^N \tilde{\Gamma}_{k,i-1}(N)$), a South step ($a\tilde{\mathcal{E}}_{k,i+1}(N-1)$), or a South-East step ($q^{N-1}\tilde{\mathcal{E}}_{k,i+1}(N-1)$). As before, for condition (iii) in the definition of the paths, the shifting is compatible with removing the first step when this is a NE or SE step. When it is a South step, the peaks are not shifted but u decreases by 1 for all peaks, so condition (iii) is still respected.

The case $i = k$, corresponding to (3.3), needs a bit more detailed explanation. The paths counted by $\tilde{\mathcal{E}}_{k,k}(N)$ begin with either an East or a North-East step. Those that begin with a North-East step where this step is deleted are the paths counted by $\tilde{\Gamma}_{k,k-1}(N)$. Shifting these one unit to the left contributes the term $q^N \tilde{\Gamma}_{k,k-1}(N)$.

For the paths that begin with an East step, first observe that the fact that every peak of coordinates $(x, k-1)$ satisfies $x - u \equiv k-1 \pmod 2$ is equivalent to the fact that every peak of coordinates $(x, k-1)$ has an even number of East steps to its left. We now consider two cases for the paths counted by $\tilde{\mathcal{E}}_{k,k}(N)$ that start with an East step where this step has been deleted. If the path does not have any other East step, then there is no peak of height $k-1$ and so we may shift the path upward, i.e., each vertex of the path (x, y) is changed to $(x, y+1)$. Shifting to the left creates a path in $\tilde{\mathcal{E}}_{k,k-1}(N)$ that does not have any vertex of the form $(x,0)$. If the path does contain another East step, then the path before the first of these other East steps is shifted up, the East step is changed to a South-East step and the rest of the path is not changed. Shifting to the left creates a path in $\tilde{\mathcal{E}}_{k,k-1}(N)$ that has at least one vertex of the form $(x,0)$. This contributes the term $q^N\tilde{\mathcal{E}}_{k,k-1}(N)$. □

It is clear that these recurrences uniquely define the series $\tilde{\mathcal{E}}_{k,i}(N)$ and $\tilde{\Gamma}_{k,i}(N)$. We may then deduce that these functions have the following nice forms:

Theorem 3.2.

$$\tilde{\mathcal{E}}_{k,i}(N)=a^N q^{\binom{N+1}{2}}(-1/a)_N\sum_{n=-N}^{N}(-1)^n\frac{q^{(k-1)n^2+(k-i)n}}{(q)_{N-n}(q)_{N+n}}, \tag{3.6}$$

$$\tilde{\Gamma}_{k,i}(N)=a^N q^{\binom{N}{2}}(-1/a)_N\sum_{n=-N}^{N-1}(-1)^n\frac{q^{(k-1)n^2+(k-i-1)n}}{(q)_{N-n-1}(q)_{N+n}}. \tag{3.7}$$

The proof is almost identical to the proof of Theorem 3.2 in [19], and hence is omitted. It uses simple algebraic manipulation to prove that these generating functions satisfy the recurrence relations of Proposition 3.1.

We now recall a lemma which may deduced from the q-Gauss summation [21] (see [19] for details of this deduction). This lemma will allow us to prove the case $X=E$ of Theorem 1.5 from Theorem 3.2.

Lemma 3.3. *For any $n\in\mathbb{Z}$, we have*

$$\sum_{N\geqslant|n|}\frac{(-aq)_n(-q^n/a)_{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}a^{N-n}}{(q)_{N+n}(q)_{N-n}}=\frac{(-aq)_\infty}{(q)_\infty}.$$

Proof of the case $X=E$ of Theorem 1.5. From (3.6), summing on N using Lemma 3.3, we obtain

$$\sum_{n,j\geqslant 0}\tilde{E}_{k,i}(n,j)a^jq^n=\frac{(-aq)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1/a)_n(-1)^na^nq^{(2k-1)\binom{n+1}{2}-in+n}}{(-aq)_n}. \tag{3.8}$$

This is Eq. (1.8) and establishes Theorem 1.5 for $X=E$. □

4. Successive ranks

In this section we prove Theorem 1.5 for $X=C$. The Frobenius representation of an overpartition [18,24] of n is a two-rowed array

$$\begin{pmatrix}a_1 & a_2 & \dots & a_N\\ b_1 & b_2 & \dots & b_N\end{pmatrix}$$

where $(a_1, \ldots, a_N)$ is a partition into distinct nonnegative parts and $(b_1, \ldots, b_N)$ is an overpartition into nonnegative parts where the first occurrence of a part can be overlined and $N + \sum(a_i + b_i) = n$.

This is called the Frobenius representation of an overpartition because there is a one-to-one correspondence between such two-rowed arrays with $N + \sum(a_i + b_i) = n$ and overpartitions of n (see [18,24]). When there are no non-overlined parts in the bottom row, we recover the Frobenius representation for ordinary partitions.

We now recall the definition of the *successive ranks* of an overpartition [19]. These are defined via the Frobenius representation and generalize Atkin's successive ranks for partitions [11].

Definition 4.1. [19] If an overpartition has Frobenius representation

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_N \\ b_1 & b_2 & \ldots & b_N \end{pmatrix}$$

then its ith successive rank r_i is $a_i - b_i$ minus the number of non-overlined parts in $\{b_{i+1}, \ldots, b_N\}$.

For example, the successive ranks of $\left(\begin{smallmatrix} 7 & 4 & 2 & 0 \\ \bar{3} & 3 & \bar{1} & 0 \end{smallmatrix}\right)$ are $(2, 0, 1, 0)$. The following proposition implies Theorem 1.5 for $X = C$.

Proposition 4.2. *There exists a one-to-one correspondence between the paths of major index n with j South steps, counted by $\tilde{E}_{k,i}(n, j)$, and the overpartitions of n with j non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i + 2, 2k - i - 2]$, counted by $\tilde{C}_{k,i}(n, j)$. This correspondence is such that the paths have N peaks if and only if the Frobenius representation of the overpartition has N columns.*

Proof. Let $\bar{E}_{k,i}(n, j)$ be the number of paths counted by $\tilde{E}_{k,i}(n, j)$ where the last condition $(x - u \equiv i - 1 \pmod 2)$ for the peaks of height $k - 1$) is dropped. In [19], we proposed a bijection between paths counted by $\bar{E}_{k,i}(n, j)$ and overpartitions of n with j non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i + 2, 2k - i - 1]$.

We now recall this map. Given a lattice path which starts at $(0, k - i)$ and a peak (x, y), let the parameter u be the number of South steps to the left of the peak. We map this peak to the column $\binom{s}{t}$, where

$$s = (x + k - i - y + u)/2,$$
$$t = (x - k + i + y - 2 - u)/2,$$

if there are an even number of East steps to the left of the peak, and

$$s = (x + k - i + y - 1 + u)/2,$$
$$t = (x - k + i - y - 1 - u)/2,$$

if there are an odd number of East steps to the left of the peak. Moreover, we overline t if the peak is a NESE peak. Starting from the right of the path, we thus construct a two-rowed array from the left. It was shown in [19] that the result is the Frobenius representation of an overpartition, that the map is reversible, and that the path has N peaks, has major index n, and has j South steps if and only if the Frobenius representation has N columns, is an overpartition of n, and has j non-overlined parts in the bottom row. Moreover, the successive rank coming from a column $\binom{s}{t}$ is $r = s - t - u$ and the conditions on the paths imply that $-i + 2 \leqslant r \leqslant 2k - i - 1$.

If we apply this map to a path counted by $\tilde{E}_{k,i}(n, j)$ then we can show that no successive rank can be equal to $2k - i - 1$. Indeed, this is equivalent to $s - t - u = 2k - i - 1$, and from the map we know that $s - t - u = k - i - y + 1$ or $k - i + y$. The first case implies that $2 - y = k$ and is therefore impossible when $k \geqslant 2$. The second case implies that $y = k - 1$ and $s = \frac{1}{2}(x + u + 2k - i - 2)$. As s is an integer, we have $x - u \equiv i \pmod 2$. But this is forbidden by the last condition of the definition of $\tilde{E}_{k,i}(n, j)$. Hence we have $\tilde{E}_{k,i}(n, j) = \tilde{C}_{k,i}(n, j)$. $\square$

5. Generalized self-conjugate overpartitions

In this section we prove Theorem 1.5 for $X = D$. We define an operation for overpartitions called k-conjugation, where $k \geqslant 2$ is an integer. From the Frobenius representation of an overpartition π, we use Algorithm III of [24] to get three partitions λ_1, λ_2 and μ as described in the following paragraph.

Let N be the number of columns of the Frobenius representation. We get λ_1, which is a partition into N nonnegative parts, by removing a staircase from the top row (i.e. we remove 0 from the smallest part, 1 from the next smallest, and so on). We get λ_2 (which is a partition into N nonnegative parts) and μ (which is a partition into distinct nonnegative parts less than N) as follows. First, we initialize λ_2 to the bottom row. Then, if the mth part of the bottom row is overlined, we remove the overlining of the mth part of λ_2, we decrease the $m - 1$ first parts of λ_2 by one and we add a part $m - 1$ to μ. For example, the overpartition whose Frobenius representation is

$$\begin{pmatrix} 7 & \overline{5} & 4 & 2 & \overline{0} \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$$

gives $\lambda_1 = (3, 2, 2, 1, 0)$, $\lambda_2 = (4, 3, 3, 2, 1)$ and $\mu = (4, 1)$.

Let λ_1' (respectively λ_2') be the conjugate of λ_1 (respectively of λ_2). Thus λ_1' and λ_2' are partitions into parts less than or equal to N. Recall that the Durfee square of a partition is the largest square contained in its diagram [9] and that the ith Durfee square is the Durfee square of the partition that is under the $(i - 1)$st Durfee square.

We now consider two regions. The first region is the portion of λ_2' below its $(k - 2)$th Durfee square (for $k = 2$, this region is λ_2'). The second region consists of the parts of λ_1' which are less than or equal to the size of the $(k - 2)$th Durfee square of λ_2' (for $k = 2$, this region is λ_1').

Definition 5.1. The *k-conjugation* consists of interchanging these two regions (if λ_2' has less than $k - 2$ Durfee squares, the k-conjugation is the identity).

Example 1. We consider the overpartition π whose Frobenius representation is

$$\begin{pmatrix} 14 & \overline{13} & 10 & \overline{8} & 7 & \overline{5} & 3 & 0 \\ 14 & \overline{12} & 10 & \overline{8} & 7 & \overline{5} & 3 & 2 \end{pmatrix}.$$

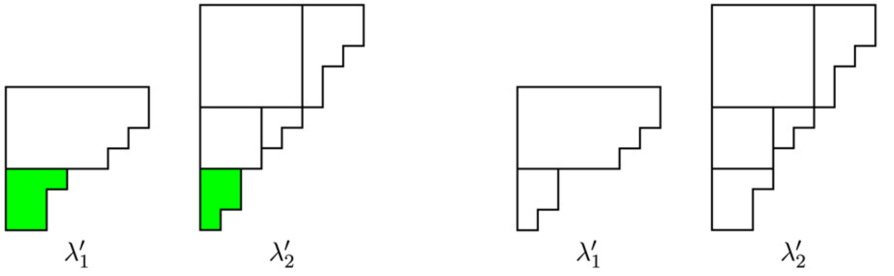

Fig. 2. Illustration of the 4-conjugation (see Example 1). For the initial overpartition π (on the left), we have $\lambda_1' = (7,7,6,5,3,2,2)$ and $\lambda_2' = (8,8,7,6,6,5,4,3,2,2,1)$. The regions highlighted are interchanged by 4-conjugation, which gives $\lambda_1' = (7,7,6,5,2,2,1)$ and $\lambda_2' = (8,8,7,6,6,5,4,3,3,2,2)$ for $\pi^{(4)}$, the 4-conjugate of π (on the right).

The above algorithm gives us $\lambda_1 = (7,7,5,4,4,3,2,0)$, $\lambda_2 = (11,10,8,7,6,5,3,2)$ and $\mu = (5,3,1)$. We thus have $\lambda_1' = (7,7,6,5,3,2,2)$ and $\lambda_2' = (8,8,7,6,6,5,4,3,2,2,1)$. λ_1' and λ_2' are represented in Fig. 2, where the two regions defined above (for $k=4$) are highlighted. If we swap these two regions, we get $\lambda_1' = (7,7,6,5,2,2,1)$ and $\lambda_2' = (8,8,7,6,6,5,4,3,3,2,2)$. Conjugating these two partitions, we have $\lambda_1 = (7,6,4,4,4,3,2,0)$ and $\lambda_2 = (11,11,9,7,6,5,3,2)$. Applying the above algorithm in reverse (remember that $\mu = (5,3,1)$), we get that the 4-conjugate of π is

$$\pi^{(4)} = \begin{pmatrix} 14 & \overline{12} & 9 & \bar{8} & 7 & \bar{5} & 3 & 0 \\ 14 & \overline{13} & 11 & \bar{8} & 7 & \bar{5} & 3 & 2 \end{pmatrix}.$$

Remark 5.2. For $k=2$, we just swap λ_1' and λ_2' (which boils down to swapping λ_1 and λ_2) and we get the F-conjugation defined by Lovejoy [24].

Remark 5.3. If there are no overlined parts, we get the k-conjugation for partitions defined by Garvan [20]. Indeed, for partitions, the $(k-2)$th Durfee square of λ_2' is in fact the $(k-1)$th Durfee square of the partition π. Consequently, the parts of λ_2' below this Durfee square (first region) are the parts of π below its $(k-1)$th Durfee square. Moreover, the parts of λ_1' which are less than or equal to the size of the $(k-2)$th Durfee square of λ_2' (second region) are the columns of π to the right of its first Durfee square whose length is less than or equal to the size of the $(k-1)$th Durfee square of π. We thus see that the regions we interchange in the k-conjugation are the same as in [20].

Definition 5.4. We say that an overpartition is *self-k-conjugate* if it is fixed by k-conjugation.

Proposition 5.5. *The generating function for self-k-conjugate overpartitions is*

$$\sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{k-1} \geqslant 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2}(-1/a)_{n_1} a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

where n_1 is the number of columns of the Frobenius symbol and $n_2, \ldots, n_{k-1}$ are the sizes of the $k-2$ first successive Durfee squares of λ_2'.

Proof. We decompose a self-k-conjugate overpartition in the following way:

- μ (region IV in Fig. 3), which is counted by
$$a^{n_1}(-1/a)_{n_1}.$$
- The staircase of the top row and the part n_1 (region III), which are counted by
$$q^{\binom{n_1+1}{2}}.$$
- The $k-2$ Durfee squares of λ_2' (region V), which are counted by
$$q^{n_2^2+\cdots+n_{k-1}^2}.$$
- The regions between the Durfee squares of λ_2' (region VI), which are counted by
$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q,$$
where
$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m(q)_{n-m}}$$
is the generating function of partitions into at most m parts less than or equal to $n-m$.
- The parts in λ_1' which are $> n_{k-1}$ and of course $\leqslant n_1$ (region I): they are counted by
$$\frac{1}{(1-q^{n_{k-1}+1})\cdots(1-q^{n_1})} = \frac{(q)_{n_{k-1}}}{(q)_{n_1}}.$$
- The two identical regions (regions II and VII), which are counted by
$$\frac{1}{(q^2;q^2)_{n_{k-1}}}.$$

Summing on $n_1, n_2, \ldots, n_{k-1}$, we get the generating function

$$\sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{k-1} \geqslant 0} (-1/a)_{n_1} a^{n_1} q^{\binom{n_1+1}{2}} q^{n_2^2+\cdots+n_{k-1}^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \frac{(q)_{n_{k-1}}}{(q)_{n_1}} \frac{1}{(q^2;q^2)_{n_{k-1}}}$$
$$= \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{k-1} \geqslant 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2}(-1/a)_{n_1} a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}. \qquad \square$$

Corollary 5.6. *When there are no overlined parts, $a \to 0$ and we get the generating function of self-k-conjugate partitions* [20].

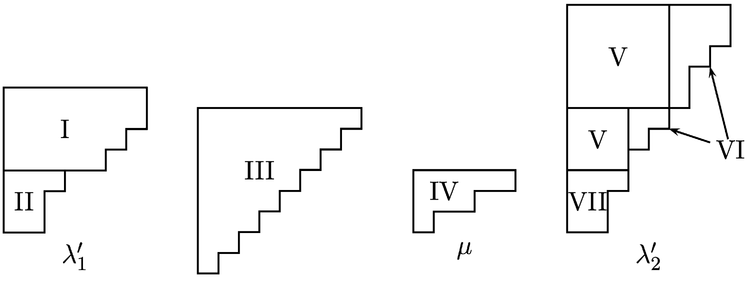

Fig. 3. Decomposition of a self-k-conjugate overpartition (in this example, $k = 4$).

Definition 5.7. Let i and k be integers with $1 \leqslant i \leqslant k$. We say that an overpartition is *self-*(k,i)*-conjugate* if it is obtained by taking a self-k-conjugate overpartition and adding a part n_j (n_j is the size of the $(j-1)$th successive Durfee square of λ_2') to λ_2' for $i \leqslant j \leqslant k-1$ (if $i = k$, no parts are added).

Remember that we denote by $\tilde{D}_{k,i}(n,j)$ the number of self-(k,i)-conjugate overpartitions with j overlined parts (or, equivalently, the number of self-(k,i)-conjugate overpartitions whose Frobenius representation has j non-overlined parts in its bottom row).

Proof of Theorem 1.5 for $X = D$. First, it is obvious from Proposition 5.5 that

$$\sum_{n,j\geqslant 0} \tilde{D}_{k,i}(n,j)a^j q^n = \sum_{n_1\geqslant n_2\geqslant\cdots\geqslant n_{k-1}\geqslant 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}.$$

To convert this multiple series into the right-hand side of (1.8), we shall use the Bailey lattice structure in [1].

Recall that a pair of sequences (α_n, β_n) form a Bailey pair with respect to a if for all $n \geqslant 0$ we have

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

In identity (3.8) of [1], let $a = q$, $\rho_1 = -1/a$, and then let n as well as all remaining ρ_i and σ_i tend to ∞. The result is that if (α_n, β_n) is a Bailey pair with respect to q, then for all $0 \leqslant i \leqslant k$ we have

$$\begin{aligned}
&\frac{1}{(q,-aq)_\infty} \times \sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 0} \frac{q^{n_1(n_1+1)/2+n_2^2+\cdots+n_k^2+n_{i+1}+\cdots+n_k}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}}\beta_{n_k} \\
&\quad = \frac{\alpha_0}{(q)_\infty^2} + \frac{1}{(q)_\infty^2}\sum_{n\geqslant 1}\frac{(-1/a)_n a^n q^{(n^2-n)(i-1)+in+n(n-1)/2}(1-q)}{(-aq)_n} \\
&\qquad \times \left(\frac{q^{(n^2+n)(k-i)}}{(1-q^{2n+1})}\alpha_n - \frac{q^{((n-1)^2+(n-1))(k-i)+2n-1}}{(1-q^{2n-1})}\alpha_{n-1}\right). \qquad (5.1)
\end{aligned}$$

Consider the Bailey pair with respect to q [26, p. 468, (E3)],

$$\beta_n = \frac{1}{(q^2;q^2)_\infty} \quad \text{and} \quad \alpha_n = \frac{(-1)^n q^{n^2}(1-q^{2n+1})}{(1-q)}.$$

Replacing n by $-n$ in the second sum, simplifying, and then replacing k by $k-1$ and i by $i-1$ gives the right-hand side of (1.8). □

6. Concluding remarks

We would like to mention that the $J_{k,i}(a;x;q)$ and $\tilde{J}_{k,i}(a;x;q)$ can be embedded in a family of functions that satisfy recurrences like those in Lemma 2.1 and are sometimes infinite products when $x=1$. For $m \geqslant 1$ we define

$$J_{k,i,m}(a;x;q) = H_{k,i,m}(a;xq;q) + axq H_{k,i-1,m}(a;xq;q), \tag{6.1}$$

where

$$H_{k,i,m}(a;x;q) = \sum_{n\geqslant 0} \frac{(-a)^n q^{kn^2+n-in-(m-1)\binom{n}{2}} x^{n(k-m-1)}(1-x^i q^{2ni})(-1/a)_n(-axq^{n+1})_\infty (x^m;q^m)_n}{(q^m;q^m)_n (x)_\infty}. \tag{6.2}$$

The case $m=1$ gives the $J_{k,i}(a;x;q)$ and $m=2$ corresponds to the $\tilde{J}_{k,i}(a;x;q)$. Equations (2.1) and (2.2) of Lemma 2.1 are true for the $H_{k,i,m}(a;x;q)$, and following the proof of (2.3), one may show that

$$H_{k,i,m}(a;x;q) - H_{k,i-m,m}(a;x;q) = x^{i-m}\left(1+x+x^2+\cdots+x^{m-1}\right) J_{k,k-i+1,m}(a;x;q).$$

It would certainly be worth investigating what kinds of combinatorial identities are stored in these general series.

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